APPENDIX 6

MATHEMATICAL TABLES

This appendix presents tabulations of (1) Fourier-transform theorems, (2) Fourier-transform pairs, (3) Hilbert-transform pairs, (4) trigonometric identities, (5) series expansions, (6) indefinite and definite integrals, (7) summations, (8) useful constants, and (9) recommended unit prefixes.

TABLE A6.1 *Fourier-Transform Theorems*

Property	*Mathematical Description*
1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$ where a and b are constants
2. Dilation (time scaling)	$g(at) \rightleftharpoons \frac{1}{\lvert a\rvert}G\left(\frac{f}{a}\right)$ where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_c t)g(t) \rightleftharpoons G(f - f_c)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)\,dt = G(0)$
7. Area under $G(f)$	$g(0) = \int_{-\infty}^{\infty} G(f)\,df$
8. Differentiation in the time domain	$\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f)$
9. Integration in the time domain	$\int_{-\infty}^{t} g(\tau)\,d\tau \rightleftharpoons \frac{1}{j2\pi f}G(f) + \frac{G(0)}{2}\delta(f)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$,
11. Multiplication in the time domain	$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f - \lambda)\,d\lambda$
12. Convolution in the time domain	$\int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau)\,d\tau \rightleftharpoons G_1(f)G_2(f)$
13. Correlation theorem	$\int_{-\infty}^{\infty} g_1(t)g_2^*(t - \tau)\,dt \rightleftharpoons G_1(f)G_2^*(f)$
14. Rayleigh's energy theorem	$\int_{-\infty}^{\infty} \lvert g(t)\rvert^2\,dt = \int_{-\infty}^{\infty} \lvert G(f)\rvert^2\,df$

TABLE A6.2 *Fourier-Transform Pairs*

Time Function	*Fourier Transform*
$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}\,(fT)$
$\text{sinc}\,(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \quad a > 0$	$\frac{1}{a + j2\pi f}$
$\exp(-a\lvert t\rvert), \quad a > 0$	$\frac{2a}{a^2 + (2\pi f)^2}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$\begin{cases} 1 - \frac{\lvert t\rvert}{T}, & \lvert t\rvert < T \\ 0, & \lvert t\rvert \geq T \end{cases}$	$T\,\text{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f - f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\,\text{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\sum_{i=-\infty}^{\infty} \delta(t - iT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$

Notes: $u(t)$ = unit step function
$\delta(t)$ = Dirac delta function
$\text{rect}(t)$ = rectangular function
$\text{sgn}(t)$ = signum function
$\text{sinc}(t)$ = sinc function

TABLE A6.3 *Hilbert-Transform Pairs*[a]

Time Function	*Hilbert Transform*
$m(t)\cos(2\pi f_c t)$	$m(t)\sin(2\pi f_c t)$
$m(t)\sin(2\pi f_c t)$	$-m(t)\cos(2\pi f_c t)$
$\cos(2\pi f_c t)$	$\sin(2\pi f_c t)$
$\sin(2\pi f_c t)$	$-\cos(2\pi f_c t)$
$\delta(t)$	$\frac{1}{\pi t}$
$\frac{1}{t}$	$-\pi\delta(t)$

[a] In the first two pairs, it is assumed that $m(t)$ is band limited to the interval $-W \leq f \leq W$, where $W < f_c$.

TABLE A6.4 *Trigonometric Identities*

$\exp(\pm j\theta) = \cos\theta \pm j\sin\theta$

$\cos\theta = \frac{1}{2}[\exp(j\theta) + \exp(-j\theta)]$

$\sin\theta = \frac{1}{2j}[\exp(j\theta) - \exp(-j\theta)]$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)]$

$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)]$

$2\sin\theta\cos\theta = \sin(2\theta)$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$

TABLE A6.5 *Series Expansions*

Taylor series	$f(x) = f(a) + \dfrac{f'(a)}{1!}(x - a) + \dfrac{f''(a)}{2!}(x - a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x - a)^n + \cdots$
where	$f^{(n)}(a) = \dfrac{d^n f(x)}{dx^n}\Big\vert_{x=a}$
MacLaurin series	$f(x) = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \cdots + \dfrac{f^{(n)}(0)}{n!}x^n + \cdots$
where	$f^{(n)}(0) = \dfrac{d^n f(x)}{dx^n}\Big\vert_{x=0}$
Binomial series	$(1 + x)^n = 1 + nx + \dfrac{n(n - 1)}{2!}x^2 + \cdots, \quad \vert nx\vert < 1$
Exponential series	$\exp x = 1 + x + \dfrac{1}{2!}x^2 + \cdots$
Logarithmic series	$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$
Trigonometric series	$\sin x = x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \cdots$
	$\cos x = 1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \cdots$
	$\tan x = x + \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 + \cdots$
	$\sin^{-1}x = x + \dfrac{1}{6}x^3 + \dfrac{3}{40}x^5 + \cdots$
	$\tan^{-1}x = x - \dfrac{1}{3}x^3 + \dfrac{1}{5}x^5 - \cdots, \quad \vert x\vert < 1$
	$\operatorname{sinc} x = 1 - \dfrac{1}{3!}(\pi x)^2 + \dfrac{1}{5!}(\pi x)^4 - \cdots$

Energy signals

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)\,dt = x(t) * x^*(-t)\big\vert_{t=\tau}$$

$$\Psi_x(f) = |X(f)|^2 = \mathcal{F}\{R_x(\tau)\}$$

$$E_x = \int_{-\infty}^{+\infty} \Psi_x(f)\,df$$

Power signals

$$R_x(\tau) = \lim_{T_0\to+\infty} \frac{1}{2T_0}\int_{-T_0}^{T_0} x(t)x^*(t-\tau)\,dt$$

$$S_x(f) = \mathcal{F}\{R_x(\tau)\}$$

$$P_x = \int_{-\infty}^{+\infty} S_x(f)\,df$$

Periodic signals

$$R_x(\tau) = \frac{1}{T}\int_0^T x(t)x^*(t-\tau)\,dt = \sum_{n=-\infty}^{+\infty} |x_n|^2 \exp(j2\pi n t/T)$$

$$S_x(f) = \sum_{n=-\infty}^{+\infty} |x_n|^2\, \delta\left(f - \frac{n}{T}\right)$$

White noise

$$S_w(f) = \frac{N_0}{2}$$

PAM signals

$$x(t) = \sum_k a_k r(t - kT)$$

$$S_x(f) = \frac{1}{T}\left[\overline{a_k^2} - (\overline{a}_k)^2\right]|R(f)|^2 + \frac{(\overline{a}_k)^2}{T^2}\sum_n \left|R\left(\frac{n}{T}\right)\right|^2 \delta\left(f - \frac{n}{T}\right)$$

If $\overline{a}_k = 0$ then $S_x(f) = \frac{1}{T}\overline{a_k^2}\,|R(f)|^2$

TABLE A6.6 *Integrals*

Indefinite integrals

$$\int x\sin(ax)\,dx = \frac{1}{a^2}[\sin(ax) - ax\cos(ax)]$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}[\cos(ax) + ax\sin(ax)]$$

$$\int x\exp(ax)\,dx = \frac{1}{a^2}\exp(ax)(ax - 1)$$

$$\int x\exp(ax^2)\,dx = \frac{1}{2a}\exp(ax^2)$$

$$\int \exp(ax)\sin(bx)\,dx = \frac{1}{a^2 + b^2}\exp(ax)[a\sin(bx) - b\cos(bx)]$$

$$\int \exp(ax)\cos(bx)\,dx = \frac{1}{a^2 + b^2}\exp(ax)[a\cos(bx) + b\sin(bx)]$$

$$\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)$$

$$\int \frac{x^2\,dx}{a^2 + b^2x^2} = \frac{x}{b^2} - \frac{a}{b^3}\tan^{-1}\left(\frac{bx}{a}\right)$$

Definite integrals

$$\int_0^\infty \frac{x\sin(ax)}{b^2 + x^2}\,dx = \frac{\pi}{2}\exp(-ab), \qquad a > 0,\ b > 0$$

$$\int_0^\infty \frac{\cos(ax)}{b^2 + x^2}\,dx = \frac{\pi}{2b}\exp(-ab), \qquad a > 0,\ b > 0$$

$$\int_0^\infty \frac{\cos(ax)}{(b^2 - x^2)^2}\,dx = \frac{\pi}{4b^3}[\sin(ab) - ab\cos(ab)], \qquad a > 0,\ b > 0$$

$$\int_0^\infty \operatorname{sinc} x\,dx = \int_0^\infty \operatorname{sinc}^2 x\,dx = \frac{1}{2}$$

$$\int_0^\infty \exp(-ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \qquad a > 0$$

$$\int_0^\infty x^2\exp(-ax^2)\,dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}, \qquad a > 0$$

TABLE A6.7 *Summations*

$$\sum_{k=1}^{K} k = \frac{K(K + 1)}{2}$$

$$\sum_{k=1}^{K} k^2 = \frac{K(K + 1)(2K + 1)}{6}$$

$$\sum_{k=1}^{K} k^3 = \frac{K^2(K + 1)^2}{4}$$

$$\sum_{k=0}^{K-1} x^k = \frac{(x^K - 1)}{x - 1}$$